Advanced Quantum Field Theory (2024/2025)
TP 2 - The harmonic oscillator at finite temperature

Exercise 1

In the lecture, the partition function $Z(\beta)$ and finite-temperature two-point correlator of the harmonic oscillator were derived using the Euclidean path integral. It is instructive to repeat these computations in the operator formalism.

1. The Hamiltonian of a harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{1}$$

   Using the ladder operators

$$a = \sqrt{\frac{m\omega}{2}}\left(q + \frac{i}{m\omega}p\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2}}\left(q - \frac{i}{m\omega}p\right) \tag{2}$$

   show that the partition function at inverse temperature β is given by

$$Z(\beta) = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}\,. \tag{3}$$

2. Compute the energy of the system given by

$$E = -\frac{\partial \ln Z(\beta)}{\partial \beta}\,. \tag{4}$$

   Determine the energy both at high and low temperatures and interpret the results.

We turn now to the thermal 2-point correlator. We first switch to Euclidean time $t = -i\tau$ and define the Euclidean time ordering by

$$T\{q(-i\tau_2)q(-i\tau_1)\} = \begin{cases} q(-i\tau_2)q(-i\tau_1) & \tau_2 > \tau_1 \\ q(-i\tau_1)q(-i\tau_2) & \tau_1 > \tau_2. \end{cases} \tag{5}$$

where $q(-i\tau)$ is an operator in the Heisenberg picture

$$q(-i\tau) = e^{iHt}|_{t=-i\tau}\, q(0)\, e^{-iHt}|_{t=-i\tau} \tag{6}$$

evaluated at Euclidean time τ. The thermal 2-point correlator is then defined by

$$\langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle_\beta = \frac{1}{Z(\beta)}\mathrm{Tr}\big(e^{-\beta H}T\{q(-i\tau_2)q(-i\tau_1)\}\big)\,. \tag{7}$$

3. Show that the thermal 2-point correlator for the harmonic oscillator is given by

$$\langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle_\beta = \frac{1}{2m\omega}\frac{\cosh(\omega(\frac{\beta}{2} - |\tau_2-\tau_1|)}{\sinh(\frac{\omega\beta}{2})}\,. \tag{8}$$

Depending on your approach, you might need the Baker–Campbell–Hausdorff formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots\,. \tag{9}$$

4. This quantity reduces to the Euclidean 2-point correlator in the zero-temperature, $\beta \to \infty$, limit. Compute the Euclidean two-point correlator for the harmonic oscillator from (8) by taking the $\beta \to \infty$ limit. You should find

$$G(\tau_2-\tau_1) \equiv \langle 0|\,T\{q(-i\tau_2)q(-i\tau_1)\}\,|0\rangle = \frac{1}{2m\omega}e^{-|\tau_2-\tau_1|\omega}\,. \tag{10}$$

Show that $G(\tau_2-\tau_1)$ obeys

$$m\left(-\frac{d^2}{d\tau^2} + \omega^2\right)G(\tau_2-\tau_1) = \delta(\tau_2-\tau_1)\,. \tag{11}$$

In other words, $G(\tau_2-\tau_1)$ is a Green function for the equations of motion of the Euclidean action. Check that also (8) is a Green function for the equations of motion. What is the difference between these two?

5. In the limit $\tau_2 - \tau_1 \to \infty, \beta \to \infty$, the connected 2-point function

$$\langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle^c_\beta \equiv \langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle_\beta - \langle q(-i\tau_2)\rangle_\beta\langle q(-i\tau_1)\rangle_\beta\,, \tag{12}$$

becomes

$$\lim_{\beta,\tau_2-\tau_1\to\infty}\langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle^c_\beta \to e^{-(E_1-E_0)(\tau_2-\tau_1)}|\,\langle 0|\,\hat{q}\,|1\rangle\,|^2, \tag{13}$$

as was discussed in the lecture. Use these results to read-off the quantities $E_1 - E_0$ and $|\,\langle 0|\,\hat{q}\,|1\rangle\,|^2$ for the harmonic oscillator from the Euclidean 2-point correlator (8).

Exercise 2

The Hamiltonian for a fermionic harmonic oscillator is given by

$$H = \omega\Big(b^\dagger b - \frac{\alpha}{2}\Big) \tag{14}$$

where $b^\dagger$ and b obey the canonical *anti*-commutation relations

$$\{b, b^\dagger\} \equiv bb^\dagger + b^\dagger b = 1 \qquad \{b^\dagger, b^\dagger\} = \{b, b\} = 0\,. \tag{15}$$

1. Compute the thermal partition function $Z(\beta)$ of the fermionic harmonic oscillator in the operator formalism. Determine the energy of the system and consider the two limits of high and low temperature.

Solutions to exercise 1

1. Using the ladder operators we can rewrite the Hamiltonian in a standard computation as

$$H = \omega\left(\frac{1}{2} + a^\dagger a\right). \tag{16}$$

The Hamiltonian is diagonalized in the usual basis for the harmonic oscillator

$$a|0\rangle = 0, \qquad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \qquad H|n\rangle = \omega\left(\frac{1}{2} + n\right)|n\rangle. \tag{17}$$

The partition function is now straightforwardly evaluated to

$$\begin{aligned} Z(\beta) = \mathrm{Tr}(e^{-\beta H}) &= \sum_{n=0}^{\infty}\langle n| e^{-\beta\frac{\omega}{2}-\beta\omega a^\dagger a}|n\rangle = e^{-\beta\frac{\omega}{2}}\sum_{n=0}^{\infty} e^{-\beta\omega n} = e^{-\beta\frac{\omega}{2}}\frac{1}{1-e^{-\beta\omega}} \\ &= \frac{1}{2\sinh(\frac{\omega\beta}{2})}. \end{aligned} \tag{18}$$

2. The energy of the system in the presence of finite temperature is given by

$$E = -\frac{\partial \ln Z(\beta)}{\partial\beta} = \partial_\beta \sinh(\frac{\beta\omega}{2}) = E_0\frac{\cosh\beta E_0}{\sinh\beta E_0}, \tag{19}$$

where we set $E_0 = \frac{\omega}{2}$ the ground state energy of harmonic oscillator.

We want to evaluate this at high and low temperatures. The dimensionless quantity to have in mind here is the ratio of temperature to characteristic energy of the harmonic oscillator, i.e., with restored units $\omega\hbar/(k_B T) = \omega\beta\hbar/k_B$. We thus find for high $\omega\beta \ll 1$ and low temperatures $\omega\beta \gg 1$

$$E \overset{\beta\omega\ll 1}{\to} E_0\frac{1}{\beta E_0} = \beta^{-1} = T \tag{20a}$$

$$E \overset{\beta\omega\gg 1}{\to} E_0\frac{e^{\beta E_0}+\ldots}{e^{\beta E_0}-\ldots} = E_0. \tag{20b}$$

The behavior in the high temperature limit follows the equipartition theorem. For low temperatures the system is expected to be in its ground state.

3. We want to compute the time-ordered two-point correlator in Euclidean signature (7) that we denote $G^{(\beta)}(\tau_2,\tau_1)$. In order to compute this quantity we are going to need the position operator in the Heisenberg picture or, equivalently, the creation and annihilation operators in the Heisenberg picture. We have

$$q(t) = e^{iHt}q(0)e^{-iHt} = \frac{1}{\sqrt{2m\omega}}(e^{iHt}a(0)e^{-iHt} + e^{iHt}a(0)^\dagger e^{-iHt}) \tag{21}$$

$$p(t) = e^{iHt}p(0)e^{-iHt} = i\sqrt{\frac{m\omega}{2}}(-e^{iHt}a(0)e^{-iHt} + e^{iHt}a(0)^\dagger e^{-iHt}) \tag{22}$$

Using the Baker–Campbell–Hausdorff formula, we can write

$$e^{iHt}a(0)e^{-iHt} = a + i[H,a] + \frac{i^2}{2!}[H,[H,a]] + \ldots = a - i\omega t a + \frac{i^2}{2!}a\omega^2 t^2 + \ldots = a e^{-i\omega t} \tag{23}$$

$$e^{iHt}a(0)^\dagger e^{-iHt} = a^\dagger e^{i\omega t} \tag{24}$$

An alternative way to derive these equations is to use the fact that the Heisenberg operators obey the classical equations of motions, so that we have

$$q(t) = q(0)\cos\omega t + \frac{p(0)}{\omega m}\sin\omega t \qquad p(t) = p(0)\cos\omega t - \omega m q(0)\sin\omega t. \tag{25}$$

Relations (23) follow then from comparison of the two sides of each equation.

Returning to the problem at hand, let us assume that $\tau_2 > \tau_1$. We can then disregard the time-ordering operator and write

$$G^{(\beta)}(\tau_2,\tau_1) = Z^{-1}\frac{1}{2m\omega}\sum_{n=0}^{\infty} e^{-\beta E_0-\beta\omega n}\langle n|\,(e^{-\omega\tau_2}a + e^{\omega\tau_2}a^\dagger)(e^{-\omega\tau_1}a + e^{\omega\tau_1}a^\dagger)\,|n\rangle \tag{26}$$

Note that only terms with both $a^\dagger$ and a survive. Using $a^\dagger a\,|n\rangle = n\,|n\rangle$ and $aa^\dagger\,|n\rangle = (n+1)\,|n\rangle$, we find

$$G^{(\beta)}(\tau_2,\tau_1) = Z^{-1}\frac{e^{-\beta E_0}}{2m\omega}\sum_{n=0}^{\infty} e^{-\beta\omega n}(e^{-\omega(\tau_2-\tau_1)}(n+1) + e^{\omega(\tau_2-\tau_1)}n)\,. \tag{27}$$

To evaluate the sums we use

$$\sum_{n=0}^{\infty} e^{-\beta\omega n}n = -\frac{1}{\omega}\sum_{n=0}^{\infty}\partial_\beta e^{-\beta\omega n} = -\frac{1}{\omega}\partial_\beta\frac{1}{1-e^{-\beta\omega}} = \frac{e^{\beta\omega}}{(e^{\beta\omega}-1)^2}\,. \tag{28}$$

Putting everything together we have

$$\begin{aligned}
G^{(\beta)}(\tau_2,\tau_1) &= Z^{-1}\frac{e^{-\beta E_0}}{m\omega}\left(\frac{e^{-\omega(\tau_2-\tau_1)}}{1-e^{-\beta\omega}} + \frac{e^{\beta\omega}(e^{-\omega(\tau_2-\tau_1)} + e^{\omega(\tau_2-\tau_1)})}{(e^{\beta\omega}-1)^2}\right) &(29)\\
&= 2\sinh(\omega\frac{\beta}{2})\frac{e^{-\beta\frac{\omega}{2}}}{2m\omega}\frac{e^{2\beta\omega-\omega(\tau_2-\tau_1)} + e^{\beta\omega+\omega(\tau_2-\tau_1)}}{(e^{\beta\omega}-1)^2} &(30)\\
&= \frac{1}{2m\omega}\frac{e^{\frac{\beta}{2}\omega-\omega(\tau_2-\tau_1)} + e^{-\frac{\beta}{2}\omega+\omega(\tau_2-\tau_1)}}{(e^{\beta\omega}-e^{-\beta\omega})} &(31)\\
&= \frac{1}{2m\omega}\frac{\cosh\omega(\frac{\beta}{2}-(\tau_2-\tau_1))}{\sinh\frac{\omega\beta}{2}}\,. &(32)
\end{aligned}$$

Remember that this was derived assuming $\tau_2 > \tau_1$. In the case $\tau_1 > \tau_2$, we would find

$$G^{(\beta)}(\tau_2,\tau_1) = \frac{1}{2m\omega}\frac{\cosh\omega(\frac{\beta}{2}-(\tau_1-\tau_2))}{\sinh\frac{\omega\beta}{2}}\,. \tag{33}$$

Since the time difference is always found to be positive we can replace it by an absolute value. Taken together we have therefore the final result

$$G^{(\beta)}(\tau_2,\tau_1) = \frac{1}{2m\omega}\frac{\cosh\omega(\frac{\beta}{2}-|\tau_2-\tau_1])}{\sinh\frac{\omega\beta}{2}}\,. \tag{34}$$

4. In the zero temperature limit $\omega\beta \gg 1$ we can write this as

$$G(\tau_2,\tau_1) = \lim_{\beta\to\infty} G^{(\beta)}(\tau_2,\tau_1) = \frac{1}{2m\omega}\frac{e^{\frac{\omega\beta}{2}}e^{-\omega|\tau_2-\tau_1|} + \dots}{e^{\frac{\omega\beta}{2}} - \dots} = \frac{1}{2m\omega}e^{-\omega|\tau_2-\tau_1|}\,. \tag{35}$$

It is straightforward to show that this is a Green function for the harmonic oscillator. Using

$$-\frac{d^2}{d\tau_2^2}e^{-\omega|\tau_2-\tau_1|} = \frac{d}{d\tau_2}(e^{-\omega|\tau_2-\tau_1|}\omega\mathrm{sgn}(\tau_2-\tau_1)) = e^{-\omega|\tau_2-\tau_1|}(-\omega^2\mathrm{sgn}(\tau_2-\tau_1)^2 + 2\omega\delta(\tau_2-\tau_1))\,. \tag{36}$$

$$m\left(-\frac{d^2}{d\tau_2^2}+\omega^2\right)G(\tau_2,\tau_1) = e^{-\omega|\tau_2-\tau_1|}\delta(\tau_2-\tau_1) = \delta(\tau_2-\tau_1) \tag{37}$$

Note that the same is true for the thermal two-point function using

$$-\frac{d^2}{d\tau_2^2}\cosh\omega(\frac{\beta}{2}-|\tau_2-\tau_1|)=\frac{d}{d\tau_2}(\omega\sinh\omega(\frac{\beta}{2}-|\tau_2-\tau_1|)\mathrm{sgn}(\tau_2-\tau_1)) \tag{38}$$

$$=-\omega^2\cosh\omega(\frac{\beta}{2}-|\tau_2-\tau_1|)\mathrm{sgn}(\tau_2-\tau_1)+2\omega\sinh\omega(\frac{\beta}{2}-|\tau_2-\tau_1|)\delta(\tau_2-\tau_1)\,. \tag{39}$$

The two functions are Green functions to the same equations of motion. However, they differ in the choice of boundary conditions.

Denoting the differential operator in (37) by $\mathcal{D}$, by definition the Green function provides a solution of the equation $\mathcal{D}q(t)=\rho(t)$ as

$$q(t)=\int d\tau G(\tau,t)\rho(\tau)\,. \tag{40}$$

Since the Green function at zero temperature vanishes at the two endpoints, $\lim_{t\to\pm\infty}G(\tau,t)=0$, it is straightforward to see that this implies $q(-\infty)=q(+\infty)=0$. It is therefore a Green function with Dirichlet boundary conditions at $t=\pm\infty$.

If we assume on the other hand that $\tau\sim\tau+\beta$, then we have

$$q(t)=\int_0^\beta d\tau G(\tau,t)\rho(\tau). \tag{41}$$

Requiring periodic boundary conditions yields the condition

$$q(0)=q(\beta)\Rightarrow\int_0^\beta d\tau G(\tau,0)\rho(\tau)=\int_0^\beta d\tau G(\tau,\beta)\rho(\tau)\Rightarrow G(\tau,0)=G(\tau,\beta) \tag{42}$$

We can check using the explicit form of $G^{(\beta)}$ (34) that this indeed the case

$$\cosh\omega(\frac{\beta}{2}-|\tau_2-\beta|)=\cosh\omega(\frac{\beta}{2}-(\beta-\tau_2))=\cosh\omega(\frac{-\beta}{2}+\tau_2)=\cosh\omega(\frac{\beta}{2}-\tau_2)=\cosh\omega(\frac{\beta}{2}-|\tau_2-0|)\,. \tag{43}$$

We can also see this periodicity a bit more generally. Consider the thermal expectation value

$$\langle A(\tau_2)B(\tau_1)\rangle_\beta=\mathrm{Tr}e^{-\beta H}T(A(\tau_2)B(\tau_1)). \tag{44}$$

Let us assume $\beta>\tau_2-\tau_1>0$. Then we can drop the time-ordering operator and find

$$\begin{aligned}\langle A(\tau_2)B(\tau_1)\rangle_\beta&=\mathrm{Tr}e^{-\beta H}T(A(\tau_2)B(\tau_1))=\mathrm{Tr}e^{-\beta H}e^{\tau_2 H}A(0)e^{-\tau_2 H}e^{\tau_1 H}B(0)e^{-\tau_1 H}\\&=\mathrm{Tr}e^{\tau_1 H}B(0)e^{-\tau_1 H}e^{-\beta H}e^{\tau_2 H}A(0)e^{-\tau_2 H}\\&=\mathrm{Tr}e^{-\beta H}e^{H(\tau_1+\beta)}B(0)e^{-H(\tau_1+\beta)}e^{\tau_2 H}A(0)e^{-\tau_2 H}\\&=\mathrm{Tr}e^{-\beta H}B(\tau_1+\beta)A(\tau_2)\\&=\mathrm{Tr}e^{-\beta H}T(B(\tau_1+\beta)A(\tau_2))\\&=\langle B(\tau_1+\beta)A(\tau_2)\rangle_\beta=\langle A(\tau_2)B(\tau_1+\beta)\rangle_\beta\end{aligned} \tag{45}$$

Here, we used cyclicity of the trace, then inserted $1=e^{-\beta H}e^{\beta H}$ and used the definition of the time-evolution operators. In going from the third-to-last line to the the second-to-last line we inserted a time-ordering symbol for free since the operators are already time-ordered. Finally, we used again the definition of the thermal correlator. Note that the ordering of the operators does not matter here, since the time-ordering operator will put them in the correct order anyways.

This condition is a (simplified) version of the KMS condition that holds generally for correlators in QFT at finite temperature.

In the present case, we have $A=B=q$ and the periodicity of $G^\beta(\tau_1,\tau_1)$ follows.

5. Let us be a bit more general and consider the analogue of (7) but for an arbitrary operator $\mathcal{O}(\tau)$. This could be any linear combination of powers of $a, a^\dagger$. Unpacking the definition, we can write (assuming again for the moment that $\tau_2 > \tau_1$)

$$\langle O(-i\tau_2)O(-i\tau_1)\rangle_\beta = Z^{-1}\sum_{n=0}^{\infty} \langle n| e^{-\beta H} e^{\tau_2 H} O(0) e^{-(\tau_2-\tau_1)H} O(0) e^{-\tau_1 H} |n\rangle \tag{46}$$

$$= Z^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle n| e^{-\beta H} e^{\tau_2 H} O(0) e^{-(\tau_2-\tau_1)H} |m\rangle \langle m| O(0) e^{-\tau_1 H} |n\rangle \tag{47}$$

$$= Z^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-E_n(\beta-(\tau_2-\tau_1))} e^{-E_m(\tau_2-\tau_1)} |\langle n| O(0) |m\rangle|^2 \tag{48}$$

If we drop the assumption of $\tau_2 > \tau_1$ we find therefore

$$\langle T\{O(-i\tau_2)O(-i\tau_1)\}\rangle_\beta = Z^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-E_n(\beta-|\tau_2-\tau_1|)} e^{-E_m|\tau_2-\tau_1|} |\langle n| O(0) |m\rangle|^2 \tag{49}$$

We see from this expression that the time-ordering operator is essential to guarantee finiteness of the correlator: without the time-ordering the correlator could grow without bound. This is a general property of Euclidean correlators. Furthermore, we see that finiteness of the correlator requires that all Euclidean times be smaller than β. This is obvious from the path-integral perspective where time is periodically identified with periodicity β.

The Euclidean one-point function for the operator $O(\tau_1)$ is given by

$$\langle O(-i\tau)\rangle_\beta = Z^{-1}\sum_{n=0}^{\infty} e^{-\beta E_n} \langle n| O(0) |n\rangle \ . \tag{50}$$

The connected 2-point function is defined as

$$\langle T\{O(-i\tau_2)O(-i\tau_1)\}\rangle^c_\beta = \langle T\{O(-i\tau_2)O(-i\tau_1)\}\rangle_\beta - \langle O(-i\tau_2)\rangle_\beta \langle O(-i\tau_1)\rangle_\beta \, . \tag{51}$$

Let us evaluate this in the large β limit. This limit projects the first sum in(49) to the ground state, i.e., the least suppressed term. Similarly, the sums in (50) reduce to the ground state contribution. Taking into account $\lim_{\beta\to\infty} Z^{-1} = e^{\beta E_0}$, we find

$$\lim_{\beta\to\infty} \langle T\{O(-i\tau_2)O(-i\tau_1)\}\rangle^c_\beta = \sum_{m=0}^{\infty} e^{E_0|\tau_2-\tau_1|} e^{-E_m|\tau_2-\tau_1|} |\langle 0| O(0) |m\rangle|^2 - |\langle 0| O(0) |0\rangle|^2 \tag{52}$$

$$= \sum_{m=1}^{\infty} e^{-(E_m-E_0)|\tau_2-\tau_1|} |\langle 0| O(0) |m\rangle|^2 \, . \tag{53}$$

Note that we used no specifics of the harmonic oscillator in deriving this result.

In the limit $\tau_2 - \tau_1 \to \infty$, the least suppressed term in this sum is

$$\lim_{\beta\to\infty} \lim_{|\tau_1-\tau_2|\to\infty} \langle T\{O(-i\tau_2)O(-i\tau_1)\}\rangle^c_\beta = e^{-(E_1-E_0)|\tau_2-\tau_1|} |\langle 0| O(0) |1\rangle|^2 \, . \tag{54}$$

In this derivation, we assumed a general operator. Inserting now $O(0) = q(0)$, we find

$$\lim_{\beta\to\infty} \lim_{|\tau_1-\tau_2|\to\infty} \langle T\{q(-i\tau_2)q(-i\tau_1)\}\rangle^c_\beta = \frac{1}{2m\omega} e^{-\omega|\tau_2-\tau_1|} \, , \tag{55}$$

the propagator at zero temperature (35). Note that, strictly speaking, for the harmonic oscillator the connected Green function coincides with the usual finite temperature Green function since the one-point function vanishes.[1]

[1]Since the question came up in the discussion: let us compare this to the usual correlators in Lorentzian zero-temperature

Solution to exercise 2

1. The Hilbert space of the fermionic harmonic oscillator is two-dimensional and spanned by

$$|0\rangle\,, |1\rangle = b^\dagger\,|0\rangle\ . \tag{58}$$

The partition function is therefore

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \langle 0|\, e^{-\beta\omega(b^\dagger b-\frac{\alpha}{2})}\,|0\rangle + \langle 1|\, e^{-\beta\omega(b^\dagger b-\frac{\alpha}{2})}\,|1\rangle = e^{\omega\beta\frac{\alpha}{2}}(1+e^{-\beta\omega}) = 2\cosh(\omega\frac{\beta}{2})e^{-\omega\beta\frac{-\alpha+1}{2}}\ . \tag{59}$$

The energy is given by

$$E = -\omega\frac{\alpha}{2} + \frac{\omega}{1+e^{\beta\omega}}\ . \tag{60}$$

For high temperatures we find $E \to \frac{-\alpha+1}{2}\omega$, for low temperatures $E \to -\frac{\alpha}{2}\omega$.

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field theory (see lecture notes section 4.1). Letting $Z[J]$ be the generating functional, then the normalized n-point correlator is defined as

$$\langle T\phi(x_1)\ldots\phi(x_n)\rangle = Z^{-1}[0](-i)^n\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_n)}Z[J]|_{J=0}\,. \tag{56}$$

This computes the n-point correlator of fields, where the denominator $Z^{-1}[0]$ ensures that we do not include contributions coming from the vacuum.

The generating functional of connected correlators is given by $W[J] = -i\ln Z[J]$. Differentiating this twice w.r.t. the source we obtain the connected two-point function.

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}W[J] = i(\langle\phi(x_1)\phi(x_2)\rangle - \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle) \tag{57}$$

The structure of the connected Green function is entirely analogous to our definition in (12).

The role of the denominator $Z[0]$ in (56) and (57) is to remove bubble diagrams from the vacuum. Note, similarly, that the role of Z^{-1} in the last exercise was to remove the "vacuum contribution" $e^{\beta E_0}$ in the large β limit.